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Spacelike hypersurfaces with constant higher order mean curvature in de Sitter space

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Abstract

In this paper we develop Minkowski-type formulae for compact spacelike immersed hypersurfaces with boundary and having some constant higher order mean curvature in de Sitter space $\mathbb{S}_1^{n+1}$. We apply them to establish a relation between the mean curvature and the geometry of the boundary, when it is a geodesic sphere contained into a horizontal hyperplane of the *steady state space* $\mathcal{H}^{n+1} \subset \mathbb{S}_1^{n+1}$.

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1. Introduction

The interest in the study of spacelike hypersurfaces immersed in spacetimes is motivated by their nice Bernstein-type properties. As for the case of de Sitter space, Goddard [5] conjectured that every complete spacelike hypersurface with constant mean curvature H in de Sitter space $\mathbb{S}_1^{n+1}$ should be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, in [1] Akutagawa showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Later, Montiel [8] solved Goddard's problem in the compact case proving that the only closed spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical hypersurfaces.

In this paper, following the ideas of Alías and Malacarne [3], we develop Minkowski-type formulae for compact spacelike hypersurfaces M^n with boundary ∂M and with constant higher order mean curvature immersed in de Sitter

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space $\mathbb{S}_1^{n+1}$ (see Proposition 1). Afterwards, considering the half $\mathcal{H}^{n+1}$ of the de Sitter space which models the so-called *steady state space*, we use these formulae to obtain our main result (Theorem 1).

For that, let a be a non-zero null vector in the past half of the null cone in the Lorentz–Minkowski space $\mathbb{L}^{n+2}$, which determines the foliation of $\mathcal{H}^{n+1}$ by the horizontal hyperplanes $L^n(\tau) = \{x \in S_1^{n+1};\ \langle x, a\rangle = \tau\}$, $\tau \in\ ]0, +\infty[$, and let $b \in \mathbb{L}^{n+2}$ be any fixed vector such that $\langle a, b\rangle \neq 0$. We prove the result below.

Theorem 1. *Let $x : M^n \to \mathcal{H}^{n+1} \subset \mathbb{S}_1^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a horizontal hyperplane $L^n(\tau)$, for a certain $\tau > 0$. Let $Y_{a,b}$ be the Killing field $\frac{1}{\langle a,b\rangle}(\langle b, .\rangle a - \langle a, .\rangle b)$ on $\mathbb{S}_1^{n+1}$. If the r-mean curvature H_r is constant for some r, $1 \le r \le n$, then*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y_{a,b}\rangle\, dS = -r \binom{n}{r} H_r \operatorname{vol}(\Omega),$$

where $T_{r-1} : \mathcal{X}(M) \to \mathcal{X}(M)$ is the $(r-1)$-Newton transformation associated with the second fundamental form of x, and Ω is the domain in $L^n(\tau)$ bounded by Σ.

Observe that the left hand side in the above formula represents the $(r-1)$-flux of the Killing field $Y_{a,b}$ on the hypersurface M, and thus Theorem 1 states that this flux does not depend on M, but only on the value of H_r and ∂M.

As an application of this result, and using an estimate of Montiel [9], we establish the following relation between the mean curvature and the geometry of the boundary.

Theorem 2. *Let $x : M^n \to \mathcal{H}^{n+1} \subset \mathbb{S}_1^{n+1}$ be a spacelike immersion of a compact hypersurface M^n with non-empty boundary ∂M in the steady state space. Suppose that M^n has constant mean curvature $H > 1$ with respect to the past-directed unit normal N and that $\partial M = S^{n-1}(b, \rho)$ is the $(n-1)$-dimensional geodesic sphere with center b and radius ρ into a horizontal hyperplane $L^n(\tau)$, for a certain $\tau > 0$. Then*

$$\rho H - \left|1 - \frac{\rho^2}{2}\right| \sqrt{H^2 - 1} \le 1.$$

As a consequence of this last result, we conclude that

> *There exists no compact spacelike hypersurface in the steady state space $\mathcal{H}^{n+1}$ with constant mean curvature $H > 1$ and spherical boundary contained in some horizontal hyperplane with radius $\sqrt{5} - 1 \le \rho \le 2$.*

From a physical point of view, the motivation for working with the spacetime $\mathcal{H}^{n+1}$ is that, in the steady state model of the universe, matter is supposed to move along geodesics normal to the hypersurfaces $L^n(\tau)$. Then, they represent constant time slices and, since all of them are isometric to a Euclidean space $\mathbb{R}^n$, in this cosmological setting the geometry of the spatial sections remains unchanged (cf. [6]).

2. Preliminaries

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz–Minkowski space ($n \ge 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{S}_1^{n+1} = \{p \in L^{n+2};\ \langle p, p\rangle = 1\}.$$

The induced metric from $\langle, \rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Moreover, if $p \in \mathbb{S}_1^{n+1}$, we can put

$$T_p(\mathbb{S}_1^{n+1}) = \{v \in \mathbb{L}^{n+2};\ \langle v, p\rangle = 0\}.$$

A smooth immersion $x : M^n \to \mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ of an n-dimensional connected manifold M^n is said to be a spacelike hypersurface if the induced metric via x is a Riemannian metric on M^n, which, as usual, is also denoted by $\langle , \rangle$.

Observe that $e_{n+2} = (0, \ldots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+2}$, which determines a time-orientation on $\mathbb{L}^{n+2}$. Thus we can choose a unique timelike unit normal field N on M^n which is past-directed on $\mathbb{L}^{n+2}$ (i.e., $\langle N, e_{n+2}\rangle > 0$), and hence we may assume that M^n is oriented by N.

Let $x : M^n \to \mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ be an immersed spacelike hypersurface in de Sitter $\mathbb{S}_1^{n+1}$, and let N be its past-directed timelike normal field. In order to set up the notation, we will denote by ∇°, $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{L}^{n+2}$, $\mathbb{S}_1^{n+1}$ and M^n, respectively. Then the Gauss and Weingarten formulae for M^n in $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ are given respectively by

$$\begin{aligned}\nabla^\circ_V W &= \overline{\nabla}_V W - \langle V, W\rangle x \\ &= \nabla_V W - \langle AV, W\rangle N - \langle V, W\rangle x\end{aligned}$$

and

$$A(V) = -\nabla^\circ_V N = -\overline{\nabla}_V N,$$

for all tangent vector fields $V, W \in \mathcal{X}(M)$, where A stands for the shape operator of M^n in $\mathbb{S}_1^{n+1}$ associated with N.

Associated with the shape operator of M there are n algebraic invariants, which are the elementary symmetric functions σ_r of its principal curvatures $\kappa_1, \ldots, \kappa_n$, given by

$$\sigma_r(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \quad 1 \le r \le n.$$

The r-mean curvature H_r of the spacelike hypersurface M is then defined by

$$\binom{n}{r} H_r = (-1)^r \sigma_r(\kappa_1, \ldots, \kappa_n) = \sigma_r(-\kappa_1, \ldots, -\kappa_n).$$

In particular, when $r = 1$,

$$H_1 = -\frac{1}{n}\sum_{i=1}^{n} \kappa_i = -\frac{1}{n}\mathrm{tr}\,(A) = H$$

is the mean curvature of M, which is the main extrinsic curvature of the hypersurface. The choice of the sign $(-1)^r$ in our definition of H_r is motivated by the fact that in that case the mean curvature vector is given by $\overrightarrow{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\overrightarrow{H}(p)$ is in the same time-orientation as $N(p)$.

Finally, we recall that a tangent vector field $Y \in \mathcal{X}(\mathbb{S}_1^{n+1})$ is said to be conformal if the Lie derivative of the Lorentzian metric $\langle , \rangle$ with respect to Y satisfies

$$\pounds_Y \langle , \rangle = 2\phi \langle , \rangle,$$

for a certain smooth function $\phi \in C^\infty(\mathbb{S}_1^{n+1})$. In other words,

$$\langle \overline{\nabla}_V Y, W\rangle + \langle V, \overline{\nabla}_W Y\rangle = 2\phi\langle V, W\rangle,$$

for all tangent vector fields $V, W \in \mathcal{X}(\mathbb{S}_1^{n+1})$. When $\phi \equiv 0$, Y is said to be a Killing vector field.

Example 1. Given any fixed vectors a and b of the Lorentz–Minkowski space $\mathbb{L}^{n+2}$ and a non-zero constant $k \in \mathbb{R}$, we consider the vector field

$$Y = k(\langle b, .\rangle a - \langle a, .\rangle b).$$

Observe that $\langle Y, x\rangle = 0$, that is, geometrically $Y(x)$ determines an orthogonal direction to the position vector x on the subspace spanned by a and b. Moreover, we easily verify that

$$\langle \overline{\nabla}_V Y, W\rangle + \langle V, \overline{\nabla}_W Y\rangle = 0,$$

for all tangent vector fields $V, W \in \mathcal{X}(\mathbb{S}_1^{n+1})$. Therefore, Y is a Killing vector field globally defined on de Sitter space.

3. The Newton transformations

In this section, we will introduce the corresponding Newton transformations

$$T_r : \mathcal{X}(M) \to \mathcal{X}(M), \quad 0 \le r \le n,$$

which arise from the shape operator A. These Newton transformations will be used in the next section to derive our Minkowski-type formulae in de Sitter space. According to our definition of the r-mean curvatures, the Newton transformations are given by

$$T_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A + \cdots + \binom{n}{1} H_1 A^{r-1} + A^r,$$

where I denotes the identity in $\mathcal{X}(M)$, or, inductively,

$$T_0 = I \quad \text{and} \quad T_r = \binom{n}{r} H_r I + A T_{r-1}.$$

Observe that the characteristic polynomial of A can be written in terms of the H_r as

$$\det(tI - A) = \sum_{r=0}^{n} \binom{n}{r} H_r t^{n-r},$$

where $H_0 = 1$. By the Cayley–Hamilton theorem, this implies that $T_n = 0$.

Besides, we have the following properties of T_r (cf. [2]).

(1) If $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on M which diagonalizes A, i.e., $AE_i = \kappa_i E_i$, $i = 1, \ldots, n$, then it also diagonalizes each T_r, and $T_r E_i = \lambda_{i,r} E_i$ with

$$\lambda_{i,r} = (-1)^r \sum_{i_1 < \cdots < i_r, i_j \neq i} \kappa_{i_1} \cdots \kappa_{i_r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} (-\kappa_{i_1}) \cdots (-\kappa_{i_r}).$$

(2) For each $1 \le r \le n-1$,

$$\operatorname{tr}(T_r) = (r+1) \binom{n}{r+1} H_r$$

and

$$\operatorname{tr}(AT_r) = -(r+1) \binom{n}{r+1} H_{r+1}.$$

(3) For every $V \in \mathcal{X}(M)$ and for each $1 \le r \le n-1$,

$$\operatorname{tr}(T_r \nabla_V A) = -\binom{n}{r+1} \langle \nabla H_{r+1}, V \rangle.$$

(4) Since $\mathbb{S}_1^{n+1}$ has constant sectional curvature, the Newton transformations T_r are divergence free, that is,

$$\operatorname{div}_M(T_r) = \operatorname{tr}(V \to (\nabla_V T_r) V) = 0.$$

4. Minkowski-type formulae in de Sitter space

In what follows, $x : M^n \to \mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ will be an immersed compact spacelike hypersurface with boundary ∂M and we will consider M^n oriented by a unit past-directed timelike normal vector field N. Furthermore, $\nu \in T_p M$ *is* the outward pointing unit conormal vector along ∂M, $\mathrm{d}M$ stands for the n-dimensional volume element of M with respect to the induced metric and the chosen orientation, and $\mathrm{d}S$ is the induced $(n-1)$-dimensional area element on ∂M.

For fixed vectors a and b of the Lorentz–Minkowski space $\mathbb{L}^{n+2}$ such that $\langle a, b \rangle \neq 0$ we will consider the particular Killing vector field $Y_{a,b} \in \mathcal{X}(\mathbb{S}_1^{n+1})$ defined by (see Example 1)

$$Y_{a,b} = \frac{1}{\langle a, b \rangle} (\langle b, .\rangle a - \langle a, .\rangle b).$$

Proposition 1 (*Minkowski-Type Formulae*). *Let* $x : M^n \to \mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ *be a spacelike immersion of a compact hypersurface with boundary* ∂M *and let* Y *be a Killing field in* $\mathbb{S}_1^{n+1}$. *If the* r*-mean curvature* H_r *of* M^n *is constant for some* r, $1 \le r \le n$, *then*
(i) $\oint_{\partial M} \langle T_{r-1}\nu, Y\rangle \, \mathrm{d}S = r\binom{n}{r} H_r \int_M \langle Y, N\rangle \, \mathrm{d}M$;
(ii) $\oint_{\partial M} \langle T_{r-1}\nu, Y_{a,b}\rangle \, \mathrm{d}S = \binom{n-1}{r-1} H_r \frac{1}{\langle a,b\rangle} \oint_{\partial M} \det(x, v_1, \ldots, v_{n-1}, a, b) \, \mathrm{d}S$,
where $\{v_1, \ldots, v_{n-1}\}$ *is a unit frame tangent to* ∂M.

Proof. (i) Denoting by $Y^T \in \mathcal{X}(M)$ the tangential component of Y and using that $\nabla_V T_r$ is self-adjoint for any $V \in \mathcal{X}(M)$ and that T_r are divergence free, we have

$$\begin{aligned}\mathrm{div}_M(T_r Y^T) &= \langle \mathrm{div}_M(T_r), Y\rangle + \sum_{i=1}^{n} \langle \nabla_{E_i} Y^T, T_r E_i\rangle \\ &= \sum_{i=1}^{n} \langle \nabla_{E_i} Y^T, T_r E_i\rangle,\end{aligned}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on M^n. Since Y is a Killing vector field, by taking the covariant derivative of $Y = Y^T - \langle Y, N\rangle N$ and using the Gauss and Weingarten formulae, we obtain that

$$\frac{1}{2}(\langle \nabla_V Y^T, W\rangle + \langle V, \nabla_W Y^T\rangle) = -\langle Y, N\rangle \langle AV, W\rangle,$$

for tangent vector fields $V, W \in \mathcal{X}(M)$. Let us choose $\{E_1, \ldots, E_n\}$ a local orthonormal frame on M diagonalizing A. Then we have

$$\langle \nabla_{E_i} Y^T, T_r E_i\rangle = \lambda_{i,r} \langle \nabla_{E_i} Y^T, E_i\rangle = \langle E_i, \nabla_{T_r E_i} Y^T\rangle.$$

Consequently,

$$\langle \nabla_{E_i} Y^T, T_r E_i\rangle = -\langle Y, N\rangle \langle A T_r E_i, E_i\rangle.$$

Therefore we obtain

$$\begin{aligned}\mathrm{div}_M(T_r Y^T) &= -\langle Y, N\rangle \mathrm{tr}\,(A T_r) \\ &= (r+1)\binom{n}{r+1} \langle Y, N\rangle H_{r+1}.\end{aligned}$$

Finally, using the divergence theorem, we conclude that

$$\begin{aligned}\oint_{\partial M} \langle T_{r-1}\nu, Y\rangle \, \mathrm{d}S &= \int_M \mathrm{div}_M(T_{r-1} Y^T) \, \mathrm{d}M \\ &= r\binom{n}{r} \int_M H_r \langle Y, N\rangle \, \mathrm{d}M.\end{aligned}$$

(ii) Let $\theta_{a,b}(v_1, \ldots, v_{n-1}) = \det(x, v_1, \ldots, v_{n-1}, a, b)$, which is an $(n-1)$-form defined in M^n. From the Gauss and Weingarten formulae, we have

$$\begin{aligned}(\nabla_Z \theta_{a,b})(X_1, \ldots, X_{n-1}) &= Z(\theta_{a,b}(X_1, \ldots, X_{n-1})) - \sum_{i=1}^{n-1} \theta_{a,b}(X_1, \ldots, \nabla_Z X_i, \ldots, X_{n-1}) \\ &= \det(Z, X_1, \ldots, X_{n-1}, a, b) \\ &\quad - \sum_{i=1}^{n-1} \langle A X_i, Z\rangle \det(x, \ldots, \underset{i}{N}, \ldots, a, b),\end{aligned}$$

for all tangent vector fields $Z, X_1, \ldots, X_{n-1} \in \mathcal{X}(M)$. Thus,

$$\mathrm{d}\theta_{a,b}(X_1, \ldots, X_n) = \sum_{i=1}^{n} (-1)^i (\nabla_{X_i} \theta_{a,b})(X_1, \ldots, \hat{X_i}, \ldots, X_n)$$

$$= \sum_{i=1}^{n}(-1)^i \det(X_i, \ldots, \overset{\wedge}{X_i}, \ldots, a, b)$$
$$- \sum_{i=1}^{n}(-1)^i \sum_{\substack{j=1\\ j\neq i}}^{n-1}\langle AX_j, X_i\rangle \det(x, \ldots, \overset{\wedge}{X_i}, \ldots, \overset{\wedge}{X_j}, \underset{j}{N}, \ldots, a, b)$$
$$= -n \det(X_1, \ldots, X_n, a, b),$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal frame on M^n that diagonalizes the shape operator A. Then, using the Gauss formula for the vectors a and b, we conclude that

$$\mathrm{d}\theta_{a,b} = n(\langle a, N\rangle\langle b, x\rangle - \langle a, x\rangle\langle b, N\rangle)\,\mathrm{d}M.$$

Therefore, from item (i) applied to the Killing vector field $Y_{a,b}$, and using the Stokes theorem, our result follows. □

Remark 1. Alías and Malacarne [3] obtained the integral formulae of the item (i) of Proposition 1 in the Lorentz–Minkowski space, while Alías, Brasil and Colares [2] obtained them in a conformally stationary spacetime, and considering the compact spacelike hypersurface M^n without boundary. The item (ii) of Proposition 1 reproduces in de Sitter space (and in a more general form) the flux formula of Alías and Pastor [4], which was used by López [7] to obtain an estimate for the height of compact spacelike surfaces with constant mean curvature in the three-dimensional Lorentz–Minkowski space $\mathbb{L}^3$.

5. The steady state space $\mathcal{H}^{n+1}$

Let $a \in \mathbb{L}^{n+2}$ be a non-zero null vector in the past half of the null cone (with vertex in the origin), that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then the open region of the de Sitter space $\mathbb{S}_1^{n+1}$ given by

$$\mathcal{H}^{n+1} = \{x \in \mathbb{S}_1^{n+1};\ \langle x, a\rangle > 0\}$$

is the so-called *steady state space* (cf. [9]).

Observe that $\mathcal{H}^{n+1}$ is extendable and, so, non-complete, being only half a de Sitter space. Its boundary, as a subset of $\mathbb{S}_1^{n+1}$, is the null hypersurface

$$\{x \in \mathbb{S}_1^{n+1};\ \langle x, a\rangle = 0\},$$

whose topology is that of $\mathbb{R} \times \mathbb{S}^{n-1}$ (cf. also [6], p. 126).

Now, we shall consider in $\mathcal{H}^{n+1}$ the timelike field

$$\mathcal{K} = a - \langle x, a\rangle x.$$

We easily see that

$$\overline{\nabla}_V \mathcal{K} = -\langle x, a\rangle V, \quad \text{for all } V \in \mathcal{X}(\mathcal{H}^{n+1}),$$

that is, $\mathcal{K}$ is a closed and conformal field on $\mathcal{H}^{n+1}$. Then (cf. [10], Proposition 1), we have that the n-dimensional distribution $\mathcal{D}$ defined on $\mathcal{H}^{n+1}$ by

$$p \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(p) = \{v \in T_p\mathcal{H}^{n+1};\ \langle \mathcal{K}(p), v\rangle = 0\}$$

determines a codimension one spacelike foliation $\mathcal{F}(\mathcal{K})$ which is oriented by $\mathcal{K}$. Moreover (cf. [8], Example 1), the leaves of $\mathcal{F}(\mathcal{K})$ are horizontal hyperplanes

$$L^n(\tau) = \{x \in S_1^{n+1};\ \langle x, a\rangle = \tau\}, \quad \tau \in\ ]0, +\infty[,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$ isometric to the Euclidean space $\mathbb{R}^n$, and having constant mean curvature one with respect to the unit past-directed normal fields

$$N_\tau(x) = \frac{1}{\tau}a - x, \quad x \in L^n(\tau).$$

Finally, we note that the hypersurfaces $L^n(\tau)$ approach the boundary of $\mathcal{H}^{n+1}$ when τ tends to zero and that when τ tends to $+\infty$ they approach the spacelike future infinity for timelike and null lines of de Sitter space.

We note that Theorem 1 reproduces in the steady state space $\mathcal{H}^{n+1}$ a corresponding integral formula in the Lorentz–Minkowski space obtained by Alías and Malacarne (cf. [3], Theorem 3). As usual, if Σ is an $(n-1)$-dimensional closed submanifold in $L^n(\tau) \hookrightarrow \mathcal{H}^{n+1}$, a spacelike hypersurface $x : M^n \to \mathcal{H}^{n+1}$ is said to be a hypersurface *with boundary* Σ if the restriction of the immersion x to the boundary ∂M is a diffeomorphism onto Σ. In this case, we identify $\partial M = \Sigma$. We now proceed to prove our main result.

Proof of Theorem 1. We first note that, for an adequate choice of the orientations on M and on Ω, we have that $M \cup \Omega$ is an n-cycle of $\mathcal{H}^{n+1}$. Thus, since $\mathcal{H}^{n+1}$ is simply connected, from the Alexander duality theorem, we have that $M \cup \Omega = \partial D$, where D is a compact oriented domain immersed in $\mathcal{H}^{n+1} \subset \mathbb{S}_1^{n+1}$ (more precisely, we may choose the orientations of M and Ω in such a way that both of them must be outward pointing vectors in relation to ∂D). On the other hand, since $Y_{a,b}$ is a Killing vector field in de Sitter space,

$$\langle \overline{\nabla}_V Y_{a,b}, V \rangle = 0, \quad \text{for all } V \in \mathcal{X}(\mathbb{S}_1^{n+1}),$$

from which it follows that

$$\mathrm{div}_{S_1^{n+1}}(Y_{a,b}) = 0.$$

In particular, this is true on $\mathcal{H}^{n+1}$. Thus, from the divergence theorem, we have

$$\int_{\partial D} \langle Y_{a,b}, \widetilde{\nu} \rangle \, \mathrm{d}S = \int_D \mathrm{div}_{S_1^{n+1}}(Y_{a,b}) = 0,$$

where $\widetilde{\nu} \in T_x D$ denotes the outward pointing unit conormal vector along ∂D and $\mathrm{d}S$ is the induced n-dimensional area element of ∂D. Then, since N and N_τ are in the same half of the null cone, we obtain that

$$\int_M \langle Y_{a,b}, N \rangle \, \mathrm{d}M - \int_\Omega \langle Y_{a,b}, N_\tau \rangle \, \mathrm{d}\Omega = 0.$$

Consequently,

$$\begin{aligned} \int_M \langle Y_{a,b}, N \rangle \, \mathrm{d}M &= \int_\Omega \langle Y_{a,b}, N_\tau \rangle \, \mathrm{d}\Omega \\ &= \int_\Omega \frac{1}{\langle a, b \rangle} (\langle b, x \rangle \langle a, N_\tau \rangle - \langle a, x \rangle \langle b, N_\tau \rangle) \, \mathrm{d}\Omega \\ &= \int_\Omega \frac{1}{\langle a, b \rangle} (-\tau \langle b, x \rangle - \tau (\frac{1}{\tau} \langle a, b \rangle - \langle b, x \rangle)) \, \mathrm{d}\Omega \\ &= -\mathrm{vol}(\Omega). \end{aligned}$$

Therefore, from item (i) of Proposition 1, we have

$$\begin{aligned} \oint_{\partial M} \langle T_{r-1}\nu, Y_{a,b} \rangle \, \mathrm{d}S &= r \binom{n}{r} H_r \int_M \langle Y_{a,b}, N \rangle \, \mathrm{d}M \\ &= -r \binom{n}{r} H_r \mathrm{vol}(\Omega). \quad \square \end{aligned}$$

Observe that in the context of Lorentzian warped products, setting $\tau = \exp(t)$, we can consider $\mathcal{H}^{n+1}$ as $-\mathbb{R} \times_{\exp(t)} \mathbb{R}^n$, which corresponds to the model for the *steady state* of the universe proposed by Bondi, Gold and Hoyle (cf. [6], p. 126). In this model, since $\mathcal{K}$ is past-directed, we have that

$$\mathcal{K}(t, p) = -\exp(t) \left(\frac{\partial}{\partial t} \right)_{(t,p)}.$$

As a consequence of Theorem 1, considering the warped product model $-\mathbb{R} \times_{\exp(t)} \mathbb{R}^n$ for $\mathcal{H}^{n+1}$, we have the following results.

Corollary 1. *Let $x : M^n \to \mathcal{H}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a horizontal hyperplane $L^n(\tau)$, for a certain $\tau > 0$. Suppose that the r-mean curvature H_r is constant for some r, $1 \le r \le n$, and that $b \in L^n(\tau)$. Then*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y_{a,b}\rangle \, \mathrm{d}S = -r \binom{n}{r} H_r \exp(nt)\,\mathrm{vol}(\varphi_t(\Omega)),$$

where $t = \ln(\tau)$, φ is the flow of $\mathcal{K}$ and Ω is the domain in $L^n(\tau)$ bounded by Σ.

Corollary 2. *Let $x : M^n \to \mathcal{H}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a horizontal hyperplane $L^n(\tau)$, for a certain $\tau > 0$. Suppose that the r-mean curvature H_r is constant for some r, $1 \le r \le n$, and that $b \in L^n(\tau)$. Then*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y_{a,b}\rangle \, \mathrm{d}S = \frac{r}{(n+1)} \binom{n}{r} H_r \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\varphi_t(\Omega))\Big|_{t=0},$$

where φ is the flow of $\mathcal{K}$ and Ω is the domain in $L^n(\tau)$ bounded by Σ.

Proof. For each $t \in \mathbb{R}$, let D_t be the domain of φ_t. Then, Ω is contained in D_t and

$$\mathrm{vol}(\varphi_t(\Omega)) = \int_{\varphi_t(\Omega)} \mathrm{d}\Omega_t = \int_{\Omega} \varphi_t^*(\mathrm{d}\Omega_t),$$

where $\mathrm{d}\Omega_t$ stands for the n-dimensional volume element of $\varphi_t(\Omega)$ with respect to the induced metric.

Because the integrand is a smooth function of t, we can differentiate this expression with respect to t by differentiating under the integral sign. Thus, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\varphi_t(\Omega))\Big|_{t=t_0} &= \int_{\Omega} \frac{\partial}{\partial t}(\varphi_t^*(\mathrm{d}\Omega_t))\Big|_{t=t_0} = \int_{\Omega} \varphi_{t_0}^*(\pounds_{\mathcal{K}}\, \mathrm{d}\Omega_{t_0}) \\ &= \int_{\Omega} \varphi_{t_0}^*(\mathrm{div}\mathcal{K}\, \mathrm{d}\Omega_{t_0}) = \int_{\varphi_{t_0}(\Omega)} \mathrm{div}\mathcal{K}\, \mathrm{d}\Omega_{t_0} \\ &= -(n+1)\exp(t_0)\mathrm{vol}(\varphi_{t_0}(\Omega)).\end{aligned}$$

Consequently, taking $t_0 = 0$, we have

$$\mathrm{vol}(\Omega) = \mathrm{vol}(\varphi_0(\Omega)) = -\frac{1}{(n+1)}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\varphi_t(\Omega))\Big|_{t=0},$$

which, from Theorem 1, finishes the proof. □

6. Proof of Theorem 2

It is straightforward to check that

$$\partial M = S^{n-1}(b,\rho) = \{x \in L^n(\tau); \langle x-b, x-b\rangle = \rho^2\}.$$

Or, equivalently,

$$\partial M = S^{n-1}(b,\rho) = \left\{x \in L^n(\tau); \langle x, b\rangle = 1 - \frac{\rho^2}{2}\right\},$$

where $\langle , \rangle$ denotes the induced metric via the inclusion $L^n(\tau) \hookrightarrow \mathcal{H}^{n+1}$. Thus, from Theorem 1 it follows that

$$\begin{aligned} nH\mathrm{vol}(B^n(\rho)) &\le \oint_{\partial M} |\langle \nu, Y_{a,b}\rangle|\, \mathrm{d}S \\ &= \oint_{\partial M} \left|\frac{1}{\langle a,b\rangle}(\langle b,x\rangle\langle a,\nu\rangle - \langle a,x\rangle\langle b,\nu\rangle)\right| \mathrm{d}S\end{aligned}$$

$$= \oint_{\partial M} \left| \frac{1}{\tau} \left(1 - \frac{\rho^2}{2}\right) \langle a, \nu\rangle - \langle b, \nu\rangle \right| \mathrm{d}S$$
$$\leq \left(\frac{1}{\tau} \left|1 - \frac{\rho^2}{2}\right| \sup_{\partial M} |\langle a, \nu\rangle| + 1 \right) \mathrm{area}(S^{n-1}(\rho))$$
$$= \left(\frac{1}{\tau} \left|1 - \frac{\rho^2}{2}\right| \sup_{\partial M} |\langle a, \nu\rangle| + 1 \right) \frac{n \,\mathrm{vol}(B^n(\rho))}{\rho}.$$

On the other hand, since $\langle a, a\rangle = 0$ and $\langle a, v\rangle = 0$ for all $v \in T_x(L^n(\tau))$, we have that

$$\langle a, \nu\rangle^2 = \langle a, N\rangle^2 - \langle a, x\rangle^2.$$

Consequently,

$$\rho H \leq \frac{1}{\tau} \left|1 - \frac{\rho^2}{2}\right| \sqrt{\sup_{\partial M} \langle a, N\rangle^2 - \tau^2} + 1.$$

Finally, since we are supposing that $H > 1$, we can use the estimate (cf. [9], Theorem 7)

$$-\tau H \leq \langle a, N\rangle < 0,$$

to conclude that

$$\rho H - \left|1 - \frac{\rho^2}{2}\right| \sqrt{H^2 - 1} \leq 1. \quad \square$$

Corollary 3. *Let $x : M^n \to \mathcal{H}^{n+1}$ be a spacelike immersion of a compact hypersurface M^n with constant mean curvature H whose boundary ∂M is a geodesic sphere of radius $\sqrt{2}$ into a horizontal hyperplane. Then*

$$|H| \leq \frac{\sqrt{2}}{2}.$$

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